Dating Problems

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1 Definitions

Throughout, take

$\mathbb{R}$ The set of real numbers

Φ The set of integrable functions $f : [0, 1] \to \mathbb{R}$ satisfying $f(x) \geq 0$ for all $x \in [0, 1]$.

2 The Good Groom

Sybil Fawlty is not very happy with her husband. She went on dates with various suitors before settling on Basil. How might she have found a better husband without going on any more dates?

Each date can go well or poorly, depending on how good of a match the suitor is for Sybil, so we characterize each suitor s by their compatiblity $c(s) \in [0, 1]$ that is the probability that a date goes well with them.

At any given time, Sybil knows the probability density function $f \in \Phi$ for the compatability $c(S)$ of a randomly chosen suitor S, the number of good dates k and the number of bad dates ℓ that she has had with the current suitor, and the number of dates q that she has left to find a groom. Though she may know a some other things (she might know the difference between a cookie and a biscuit, or she might remember her dating history with rejected suitors) but none of that information will help her choose a good groom.

After each date, Sybil must decide, based on what she knows, whether to invite the same suitor on another date, or reject them and ask the next suitor instead. A rejected suitor will be offended, so Sybil can't see them anymore. At the end of all her dates, Sybil will marry a suitor G who she has not yet rejected. Sybil's plan is illustrated in Figure 1, but she is still missing the rule she should use to decide when to reject a suitor.

Problem 1 (The Good Groom Problem)**.** When should Sybil reject a suitor in order to maximize the expected compatibility $c(G)$ of her groom?

Let R be the event that Sybil rejects her current suitor and Q be the event that the next date goes well. Sybil should reject the current suitor whenever

$$E[c(G)|R] > E[c(G)|\bar{R}]$$

Suppose that Sybil chooses optimally whether or not to reject a suitor. If she hasn't gone on any dates with the current suitor, then

$$E[c(G)] = E[c(G)|R] = E[c(G)|\bar{R}]$$

since all suitors share the same prior distiribution for compatibility. Otherwise,

$$E[c(G)] = \max(E[c(G)|R], E[c(G)|\bar{R}])$$

since Sybil will only reject if it increases $E[c(G)]$.

Using conditional probabilities,

$$E[c(G)|\bar{R}] = \Pr(Q|\bar{R})E[c(G)|\bar{R} \cap Q] + \Pr(\bar{Q}|\bar{R})E[c(G)|\bar{R} \cap Q]$$

Whenever a date with a particular suitor goes well, Sybil learns about that suitors compatiblity. Specifically, a good date transforms the compatibility PDF for that suitor from f to Kf, where

$$(Kf)(x) = \frac{xf(x)}{\int tf(t)dt}.$$

Using induction we show that the PDF of the compatibility of a suitor after k good dates is

$$(K^k f)(x) = \frac{x(K^{k-1}f)(x)}{\int t(K^{k-1}f)(t)dt} = \frac{x\frac{x^{k-1}f(x)}{\int t^{k-1}f(t)dt}}{\int t\frac{t^{k-1}f(t)}{\int s^{k-1}f(s)ds}dt} = \frac{x^k f(x)}{\int t^k f(t)dt}$$

Similarly, a bad date transforms the compatibility PDF for that suitor from f to Lf, where

$$(Lf)(x) = \frac{(1-x)f(x)}{\int (1-t)f(t)dt}.$$

Then the PDF of the compatibility of a suitor after ℓ bad dates is

$$(L^\ell f)(x) = \frac{(1-x)^\ell f(x)}{\int (1-t)^\ell f(t)dt}.$$

Then the PDF for the compatibility of the current suitor is $K^k L^\ell f$ where

$$(K^k L^\ell f)(x) = \frac{x^k(1-x)^\ell f(x)}{\int t^k(1-t)^\ell f(t)dt},$$

because K and L commute.

This allows us to characterize the probability $\mu_f(k,\ell)$ of the next date with the same suitor going well and the probability $\bar{\mu}_f(k,\ell)$ of the next date with the same suitor going poorly:

$$\begin{aligned}\Pr(Q|\bar{R}) &= \mu_f(k,\ell) = \int x \cdot (K^k L^\ell f)(x)dx \\ \Pr(\bar{Q}|\bar{R}) &= \bar{\mu}_f(k,\ell) = 1 - \mu(k,\ell)\end{aligned}$$

Then we can characterize $E[c(G)]$ completely in terms of f, q, k, ℓ.

Define

$$S_f(q,k,\ell) = \begin{cases} \mu_f(k,\ell)S_f(q-1,k+1,\ell) + \bar{\mu}_f(k,\ell)S_f(q-1,k,\ell+1) & k+\ell = 0 \\ \max(\mu_f(k,\ell), \mu_f(0,0)) & q = 0, k+\ell > 0 \\ \max(S_f(q,0,0), \mu_f(k,\ell)S_f(q-1,k+1,\ell) + \bar{\mu}_f(k,\ell)S_f(q-1,k,\ell+1)) & (k+\ell)q > 0 \end{cases}$$

If Sybil has not been on any dates with the current suitor, then $k+\ell = 0$, so

$$\begin{aligned}E[c(G)] &= E[c(G)|\bar{R}] \\ &= \Pr(Q|\bar{R})E[c(G)|Q \cap \bar{R}] + \Pr(\bar{Q}|\bar{R})E[c(G)|\bar{Q} \cap \bar{R}] \\ &= \mu_f(k,\ell)S_f(q-1,k+1,\ell) + \bar{\mu}_f(k,\ell)S_f(q-1,k,\ell+1) \\ &= S_f(q,k,\ell)\end{aligned}$$

If Sybil has no dates left, then $q = 0$, so

$$\begin{aligned} E[c(G)] &= \max(E[c(G)|R], E[c(G)|\bar{R}]) \\ &= \max(\Pr(Q|R), \Pr(Q|\bar{R})) \\ &= \max(\mu_f(0,0), \mu_f(k,\ell)) \\ &= S_f(q,k,\ell) \end{aligned}$$

Othwerwise, $(k+\ell)q > 0$, so

$$\begin{aligned} E[c(G)] &= \max(E[c(G)|R], E[c(G)|\bar{R}]) \\ &= \max(E[c(G)|R], \Pr(Q|\bar{R})E[c(G)|Q \cap \bar{R}] + \Pr(\bar{Q}|\bar{R})E[c(G)|\bar{Q} \cap \bar{R}] \\ &= \max(S_f(q,0,0), \mu_f(k,\ell)S_f(q-1,k+1,\ell) + \bar{\mu}_f(k,\ell)S_f(q-1,k,\ell+1)) \\ &= S_f(q,k,\ell) \end{aligned}$$

Thus we have our answer: Sybil should reject the current suitor whenever

$$S_f(q,0,0) > \mu_f(k,\ell)S_f(q-1,k+1,\ell) + \bar{\mu}_f(k,\ell)S_f(q-1,k,\ell+1)).$$

3 Further Questions

Problem 2 (The Good Time Problem)**.** What rule should Sybil use if she wants to maximize the number of good dates during the process, rather than the compatibility of the suitor she finally chooses?

Problem 3. What rule should Sybil use if she does not know the prior distribution f of the compatibility of the suitors?

Problem 4. What rule should Sybil use if the outcome of a date is distributed in $[0,1]$ rather than $\{0,1\}$?

4 Reference

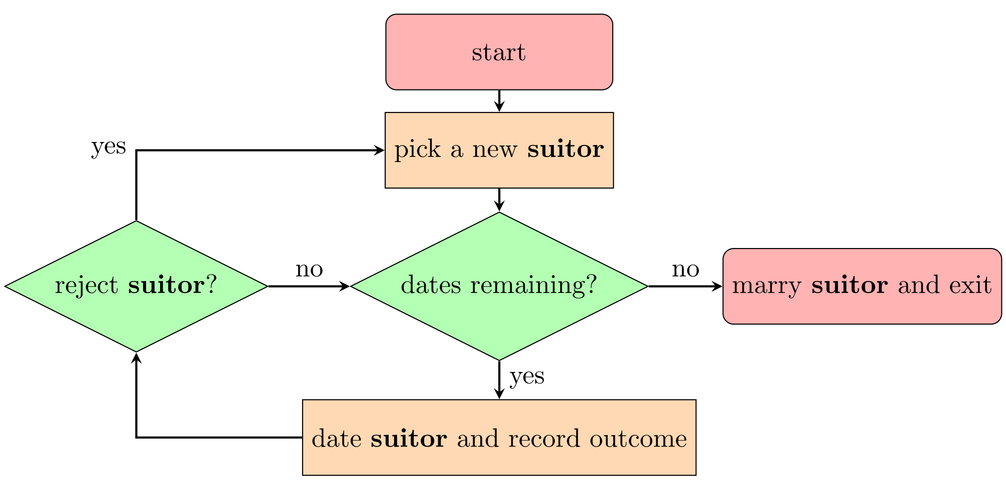

Figure 1: Sybil's Dating Process