Using symmetries for electric and magnetic fields
PHY 202, March 2004

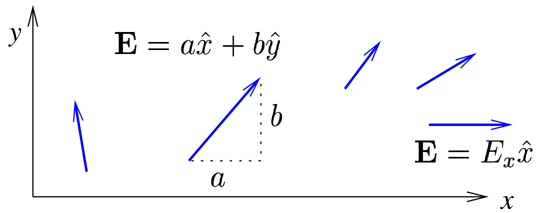
Since symmetries are not clearly discussed in most textbooks, here are some notes discussing their use in the case of electric and magnetic fields.

1 The Electric Field

Since the electric field $\mathbf{E}$ is a vector, it has three components:

$$\mathbf{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \ . \tag{1}$$

In addition, the electric field is also a function of position: I can measure it at various points in space (x, y, z). Thus, $\mathbf{E}$ and its components are functions of x, y, and z.



The basic rule for symmetries of the electric field can be obtained from Coulomb's law and the superposition principle:

$\mathbf{E}$ has the same symmetries as the charges that produce it.

For translational and rotational symmetries, this law is pretty easy to apply. However, the application of this rule to reflectional symmetry is a bit tricky.

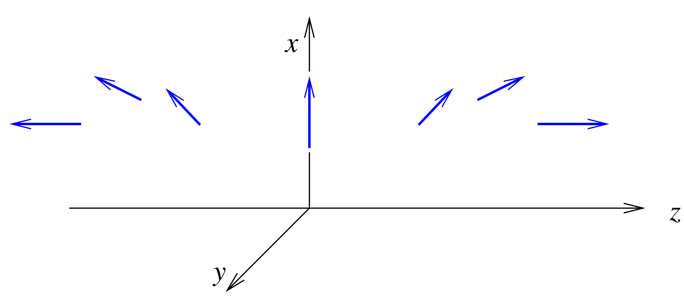
Let us say that the electric field is *symmetric* under the reflection $z \to -z$. This means that the transformed field

$$\mathbf{E} \to E_x(x, y, -z)\hat{x} + E(x, y, -z)\hat{y} - E_z(x, y, -z)\hat{z} \tag{2}$$

must be equal to the original field, Equation (1). In terms of the components of $\mathbf{E}$, this means that

$$E_x(x, y, z) = E_x(x, y, -z) \ ,$$
$$E_y(x, y, z) = E_y(x, y, -z) \ , \text{ and}$$
$$E_z(x, y, z) = -E_z(x, y, -z) \ .$$

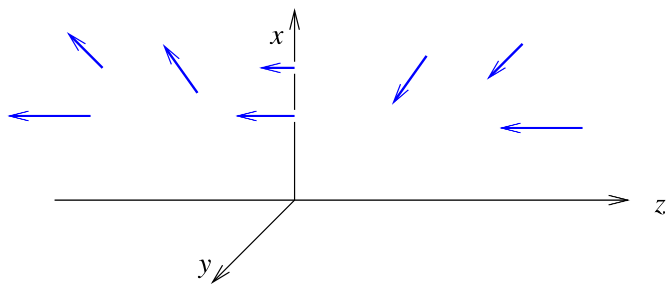
Furthermore, the third equation implies that the z-component of the field is zero in the plane of reflection $E_z(x, y, 0) = 0$. That is, in the symmetry plane $z = 0$, the electric field can not have a component perpendicular to that plane.



If the electric field is *antisymmetric* under the reflection $z \to -z$, the transformed field (2) must be equal to *minus* the original field (1). In terms of the components of $\mathbf{E}$, this means that

$$\begin{aligned} E_x(x,y,z) &= -E_x(x,y,-z) \ , \\ E_y(x,y,z) &= -E_y(x,y,-z) \ , \ \text{and} \\ E_z(x,y,z) &= E_z(x,y,-z) \ . \end{aligned}$$

Furthermore, the first two equations imply that $E_x(x,y,0) = E_y(x,y,0) = 0$. That is, in the symmetry plane $z = 0$, the electric field must be perpendicular to that plane.



An electric dipole is a good example of a system that is antisymmetric under a reflection.

2 The Magnetic Field

As with the electric field, the magnetic field $\mathbf{B}$ is a vector with three components:

$$\mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \ . \tag{3}$$

The basic rule for symmetries of the magnetic field is:

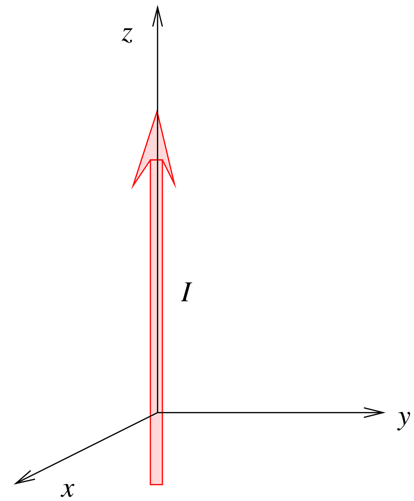
> **B** has the same translation and rotation symmetries as the currents that produce it.

Thus, for translations and rotations, we have the same rule that we had for the electric field. But what about reflections? The rule, which comes from the Biot-Savert law, states that:

> **B** has the *opposite* reflection symmetries as the currents that produce it.

Application of this rule is confusing enough that it is usually easier to use the Biot-Savert law or the "other right hand rule" to figure out what is going on.

In case you are interested in how this works, let us consider a long, straight wire placed on the z-axis which is carrying a current I.



Since the current has a direction, the current is antisymmetric under the reflection $z \to -z$. Using the rule for reflections of the magnetic field, $\mathbf{B}$ must be symmetric under $z \to -z$. In terms of the components of $\mathbf{B}$, this means that

$$\begin{aligned} B_x(x,y,z) &= B_x(x,y,-z) \ , \\ B_y(x,y,z) &= B_y(x,y,-z) \ , \text{ and} \\ B_z(x,y,z) &= -B_z(x,y,-z) \ . \end{aligned}$$

In particular, this implies that $B_z(x,y,0) = 0$. Since we also have translational symmetry $z \to z+c$, this means that the z-component of $\mathbf{B}$ must be zero everywhere.

On the other hand, the current is symmetric under reflections $y \to -y$. This implies that $\mathbf{B}$ must be antisymmetric under $y \to -y$. In terms of the components of $\mathbf{B}$, we have

$$\begin{aligned} B_x(x,y,z) &= -B_x(x,-y,z) \ , \\ B_y(x,y,z) &= B_y(x,-y,z) \ , \text{ and} \\ B_z(x,y,z) &= -B_z(x,-y,z) \ . \end{aligned}$$

In particular, $B_x(x,0,z) = 0$ and $B_z(x,0,z) = 0$. Thus, for points on the x-axis, $\mathbf{B}$ must point in the $\pm y$ direction! By rotational symmetry, the field must always point in the "angular direction." This result is in agreement with the "other right hand rule."