PHY 201 Homework 10

Due Friday, November 22 at noon.

1. The solution of the differential equation $y'' + \omega^2 y = 0$ can be expressed in three different ways:

$$
\begin{aligned}
y_1(t) &= A_1 e^{i\omega t} + A_2 e^{-i\omega t} \\
y_2(t) &= B_1 \cos(\omega t) + B_2 \sin(\omega t) \\
y_3(t) &= F \cos(\omega t + \phi)\ .
\end{aligned}
$$

   In this problem, we will show that these solutions are equivalent to each other. In other words, $y_1(t) = y_2(t) = y_3(t)$ for all values of t.

   (a) Use trig identities to rewrite the equation $y_3(x)$ so that it is in the form:

$$y_3(t) = (\text{something}) \cos(\omega t) + (\text{something else}) \sin(\omega t)$$

   (b) Now, you want to compare your result with $y_2(t)$. To show that your expression is equal to $y_2(t)$ for all values of t, we first choose $\omega t = 0$. Plug this into your expression and into $y_2(t)$. This will yield B_1 as a function of F and ϕ. Do the same for the choice $\omega t = \pi/2$.

   (c) Now, do the same for $y_1(t)$: expand the exponentials using the Euler identity and express y_1 in terms of sines and cosines. Now, assume $y_1(t) = y_2(t)$ for all t, how are A_1 and A_2 related to B_1 and B_2?

2. Consider a mass m hanging from a spring with spring constant k. Let $y(t)$ be the displacement of the mass from its equilibrium position.

   (a) Write down the "equation of motion" $\mathbf{F} = m\mathbf{a}$ for this problem. First consider what happens when the mass is sitting motionless at equilibrium, $y(t) = 0$; then consider oscillations with nonzero amplitude, $y(t) \neq 0$.

   (b) What are the symmetries of this system? What is conserved?

   (c) Let us make the *Ansatz* that $y(t) = C \sin(\omega t)$. (*Ansatz* is German for 'basic approach' or 'starting assumption.') Find the velocity and acceleration as a function of time by taking the appropriate derivatives of $y(t)$.

   (d) Verify that our *Ansatz* is indeed a solution of the equation of motion.

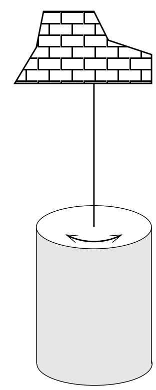
   (e) Find ω as a function k and m.

   (f) Find the kinetic energy as a function of time.

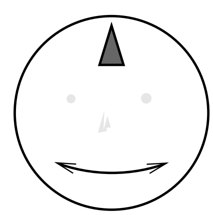
   (g) Find the potential energy of the spring as a function of time.

   (h) How does the sum of the potential and kinetic energy behave as a function of time? Does your answer agree with your answer to 2b?

3. A 100 g mass hangs on the end of a Hooke's law spring suspended vertically. When 40 g is added to the mass, the spring hangs an additional 5 cm. The mass is sent into oscillation with an amplitude of 6 cm.
   (a) What is the period of oscillation?
   (b) How long does it take for the mass to travel from the equilibrium position to a position of maximum amplitude?
   (c) What is the maximum velocity of the mass?



4. A uniform, solid cylinder of mass 5 kg and radius 4 cm is suspended as a torsion pendulum by a steel wire. The torsion constant for the wire is $\kappa = 4 \cdot 10^{-4}\,\frac{\mathrm{N \cdot m}}{\mathrm{rad}}$. Find the equation of motion for this system, make an *Ansatz* for the solution, and find the period of oscillation.
   The torsion constant is the rotational analog of the spring constant; instead of $F = -kx$, we have $\tau = -\kappa\alpha$.



5. A thin hoop of radius $r = 20\,\mathrm{cm}$ is resting on a knife-edge so that it can swing back and forth in the plane of the hoop.
   (a) Derive the equation of motion.
   (b) Find the small amplitude limit of the equation of motion.
   (c) Use this to find the period for small-amplitude oscillations.
   (d) What is the length of a simple pendulum that has the same period as the hoop?

   To solve this problem, you will need to read about the "physical pendulum" and "the parallel axis theorem" in your textbook.