The Lorentz group and the transverse lattice

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1 Abstract

This paper focuses primarily on derivations for generators of the Lorentz group in both the continuum and for the transverse lattice using the commutation relations of Section 2. It will be shown that while the lattice approximations of P^- and P^r can be continually improved by adding extra terms to their Taylor expansions, the lattice version of M_{-r} is exact. This paper also considers simple models for mesons and baryons in order to observe motion in the transverse directions.

2 Notation and Generators

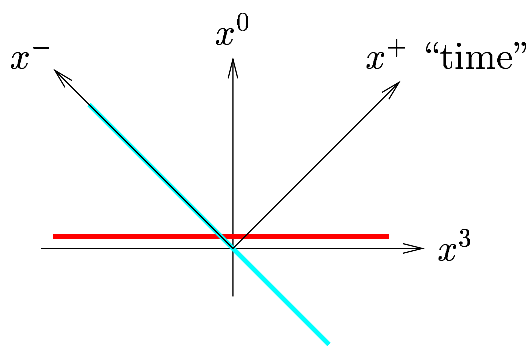

Figure 1: Light front coordinates. P^+ generates translation of the quantization surface in the x^- direction. P^- generates translation of the quantization surface in the x^+ direction.

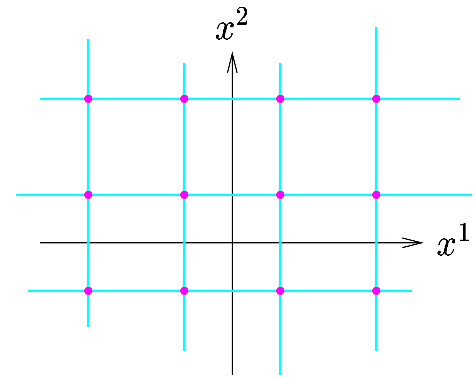

Figure 2: Transverse lattice.

This section will review some of the notation used in light-front field theory and discuss the generators of the Poincaré group. Let us introduce the coordinates

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}} \tag{1}$$

where x^+ is light-front "time" and x^- is the "longitudinal" spatial coordinate. A theory is quantized on a surface of constant x^+. The "transverse" spatial coordinates are simply

$$\mathbf{x} = \left(x^1, x^2\right) \ . \tag{2}$$

Thus, $a^+ = a_-$ and inner products are written as

$$a \cdot b = a^\mu b_\mu = a^+ b^- + a^- b^+ - \mathbf{a} \cdot \mathbf{b} \ . \tag{3}$$

Generally, transverse components are labeled by lower case Latin indices: $a_k = -a^k$ and $\mathbf{a} \cdot \mathbf{b} = a_k b_k$.

Many of the properties of light-front field theory can be seen by looking at the generators of the Poincaré group. The ten generators P^μ and $M^{\mu\nu} = -M^{\nu\mu}$ obey the algebra

$$\left[P^\mu, P^\nu\right] = 0 \tag{4}$$

$$\left[M^{\mu\nu}, P^\rho\right] = i\left(g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu\right) \tag{5}$$

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right) \ . \tag{6}$$

The three dynamical generators are:

$\mathbf{P^-}$ Generates of translations of the quantization surface in the x^+ direction; it is the light-front Hamiltonian. The energy of a free particle of mass m is

$$P^- = \frac{\mathbf{P}^2 + m^2}{2P^+} \quad . \tag{7}$$

$\mathbf{M_{+r}}$ Generates rotations of the quantization surface $x^+ = 0$ about the cone $x_\mu x^\mu = 0$.

$$\left[M_{+r}, P^-\right] = 0 \tag{8}$$

The seven kinematic generators are:

$\mathbf{P^+}$ Longitudinal momentum (x^- direction). Causality implies $P^\mu P_\mu > 0$, $P^0 > 0$. Thus, P^+ has a positive definite spectrum for massive particles.

$$\left[M_{+r}, P^+\right] = -iP_r \quad \left[P^+, P^-\right] = 0 \tag{9}$$

$\mathbf{P^r}$ Transverse momentum (x^r direction).

$$\left[M_{+r}, P^s\right] = ig_r{}^s P^- \quad \left[P^r, P^-\right] = \left[P^r, P^+\right] = 0 \tag{10}$$

$\mathbf{J_3 = M_{12}}$ Generates rotations in the x^1, x^2 plane. Light-front "helicity."

$$\begin{aligned} \left[J_3, M_{+r}\right] &= -i\epsilon_r{}^s M_{+s} \quad \left[J_3, P^-\right] = \left[J_3, P^+\right] = 0 \\ \left[J_3, P^r\right] &= -i\epsilon^r{}_s P^s \end{aligned} \tag{11}$$

where ϵ_{rs} is the antisymmetric tensor $\epsilon_{12} = 1$.

$\mathbf{M_{-r}}$ Transverse boost generator.

$$\begin{aligned} \left[P^-, M_{-r}\right] &= iP_r \quad \left[M_{-r}, P^+\right] = 0 \\ \left[M_{+r}, M_{-s}\right] &= -i\left(\epsilon_{rs} J_3 - g_{rs} K_3\right) \\ \left[P^r, M_{-s}\right] &= -ig^r{}_s P^+ \\ \left[J_3, M_{-r}\right] &= -i\epsilon_r{}^s M_{-s} \end{aligned} \tag{12}$$

$\mathbf{K_3 = M_{-+}}$ Longitudinal boost generator. It simply rescales the other generators.

$$\begin{aligned} \left[K_3, P^-\right] &= -iP^- \quad \left[K_3, P^r\right] = \left[K_3, J_3\right] = 0 \\ \left[K_3, M_{+r}\right] &= -iM_{+r} \\ \left[K_3, P^+\right] &= iP^+ \\ \left[K_3, M_{-r}\right] &= iM_{-r} \end{aligned} \tag{13}$$

3 The Longitudinal Boost Operator: K_3

In the following section, I will derive the longitudinal boost operator for light front coordinates, K_3. In this section, transverse coordinates will be suppressed in order to simplify notation. The longitudinal boost operator can be constructed by first defining an eigenstate of the longitudinal momentum operator,

$$P^+|\psi(p^+)\rangle = p^+|\psi(p^+)\rangle. \tag{14}$$

Defining a boosted state,

$$|\psi'\rangle = e^{iK_3\eta}|\psi(p^+)\rangle, \tag{15}$$

measuring the momentum on this state, and Taylor expanding in η, from the algebra of Equation (13),

$$\begin{aligned} P^{+}|\psi'\rangle &= P^{+}\left(1+iK_3\eta+\cdots\right)|\psi(p^{+})\rangle \\ &= \left(P^{+}+i\left(K_3P^{+}-iP^{+}\right)\eta+\cdots\right) \\ &= e^{i(K_3-i)\eta}P^{+}|\psi(p^{+})\rangle \\ &= e^{\eta}p^{+}|\psi'\rangle. \end{aligned} \tag{16}$$

Thus, boosting a state increases its longitudinal momentum by a factor of e^{η}.

The K_3 operator can be expressed in terms of the creation and annihilation operators, $a^{\dagger}(k)$ and $a(k)$, where k represents the variable for longitudinal momentum. We assume the usual commutation relations for these operators:

$$\begin{aligned} \left[a(k), a^{\dagger}(k')\right] &= \delta(k-k') \\ \left[a(k), a(k')\right] &= 0 \\ \left[P^{+}, a^{\dagger}(k)\right] &= k\, a^{\dagger}(k). \end{aligned} \tag{17}$$

Clearly, P^{+} has the form,

$$P^{+} = \int dk\, k\, a^{\dagger}(k)a(k). \tag{18}$$

In terms of these operators, the expression derived in Equation (16) becomes,

$$e^{iK_3\eta}a^{\dagger}(k)|0\rangle = e^{\eta/2}a^{\dagger}(e^{\eta}k)|0\rangle. \tag{19}$$

Where $e^{\eta/2}$ is the normalization factor of the boosted state[1]. This result can be used to find the commutator relations

$$[a(k), K_3] = i\left[k\frac{da(k)}{dk} + \frac{a(k)}{2}\right] \tag{20}$$

$$[a^{\dagger}(k), K_3] = i\left[k\frac{da^{\dagger}(k)}{dk} + \frac{a^{\dagger}(k)}{2}\right]. \tag{21}$$

One can show that the following *ansatz* for the value of K_3 satisfies these commutator relations:

$$K_3 = \frac{i}{2}\int dk\, k\left[a^{\dagger}(k)\frac{da(k)}{dk} - \left(\frac{da^{\dagger}(k)}{dk}\right)a(k)\right]. \tag{22}$$

4 Operators in the Transverse Continuum

4.1 Revising Creation and Annihilation Operators

This section will examine operators that act in the transverse directions. Let $\mathbf{x}$ represent position and $\mathbf{p}$ represent momentum in the transverse directions. In this section, the coordinates $\mathbf{x}$ are continuous. (We will introduce the transverse lattice in Chapter 5.) Creation and annihilation operators will take on the forms $a^{\dagger}(k,\mathbf{x})$ and $a(k,\mathbf{x})$. The following commutation relations hold for these operators:

$$\begin{aligned} \left[a(k,\mathbf{x}), a^{\dagger}(k',\mathbf{y})\right] &= \delta(k-k')\delta^2(\mathbf{x}-\mathbf{y}) \\ \left[a(k,\mathbf{x}), a(k',\mathbf{y})\right] &= 0 \\ \left[P^{+}, a^{\dagger}(k,\mathbf{x})\right] &= k\, a^{\dagger}(k,\mathbf{x}). \end{aligned} \tag{23}$$

Derivation of the operator P^{r}[2] is accomplished by Fourier transforming from momentum space to position space. One can verify that P^{r} is,

$$P^{r} = \frac{i}{2}\int dk\, d^2\mathbf{x}\left(\frac{\partial a^{\dagger}(k,\mathbf{x})}{\partial \mathbf{x}}a(k,\mathbf{x}) - a^{\dagger}(k,\mathbf{x})\frac{\partial a(k,\mathbf{x})}{\partial \mathbf{x}}\right). \tag{24}$$

[1] This normalization factor is consistent with the fact that the commutator

$$\left[a(k), a^{\dagger}(k')\right] = \delta(k-k')$$

is invariant under boosts.

[2] $r = 1, 2$ denotes the direction in which an operator acts.

4.2 The Transverse Boost Operator: M_{-r}

The transverse boost operator, M_{-r}, was constructed by first defining an eigenstate of the P^r operator,

$$P^r|\psi\rangle = p^r|\psi\rangle. \tag{25}$$

Boosting this state gives

$$|\psi'\rangle = e^{iM_{-r}\eta}|\psi\rangle. \tag{26}$$

From Equation (12), measuring the momentum of the boosted state gives

$$\begin{aligned} P^r|\psi'\rangle &= P^r\left(1 + iM_{-r}\eta + \frac{(iM_{-r}\eta)^2}{2!} + \cdots\right)|\psi\rangle \\ &= \left(P^r + i\eta\left(M_{-r}P^r - iP^+\right) + \frac{(i\eta)^2}{2!}\left(M_{-r}P^r - iP^+\right)M_{-r} + \cdots\right)|\psi\rangle \\ &= e^{iM_{-r}\eta}P^r|\psi\rangle + \eta\, e^{iM_{-r}\eta}P^+|\psi\rangle \end{aligned} \tag{27}$$

Thus,

$$P^r|\psi'\rangle = (p^r + \eta k)|\psi'\rangle. \tag{28}$$

To express this in terms of creation and annihilation operators, consider the effect of M_{-r} in position space. Boosting the state

$$|k, p^r\rangle = \int d^2\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k, \mathbf{x})|0\rangle \tag{29}$$

will cause the transverse momentum to follow the algebra derived above (Equation 3.6). Thus,

$$e^{iM_{-r}\eta}|k, p^r\rangle = |k, p^r + \eta k\rangle. \tag{30}$$

Translating this relation into position space (by way of Fourier transforms) gives,

$$e^{iM_{-r}\eta}a^\dagger(k, \mathbf{x})|0\rangle = e^{ix^r\eta k}a^\dagger(k, \mathbf{x})|0\rangle. \tag{31}$$

The boosted state gains only a phase factor in position space. Thus, M_{-r} operator was found to be,

$$M_{-r} = \int dk\, d^2\mathbf{x}\ k\ \mathbf{x}\ a^\dagger(k, \mathbf{x})\ a(k, \mathbf{x}). \tag{32}$$

Testing this solution against the fundamental relations of Equation (12) gave the correct commutators. This is done with the assumption that the surface term aquired from integration by parts,

$$\int d^2\mathbf{x}\ k\ \mathbf{x}\ a^\dagger(k, x)\ a(k, x)\Big|_{k=-\infty}^{\infty}, \tag{33}$$

can be removed with the assumption that any state it encounters must go to zero at $k = 0, \infty$.

5 Operators in the Transverse Lattice

5.1 P^r in the Transverse Lattice

Translating operators derived for the continuum into the transverse lattice causes the continuous variable x to become the discrete variable x_i. Thus, $x_{i+1} - x_i = a$, where a is the lattice spacing. The momentum operator for the transverse lattice, P^r, was found by considering the discrete versions of momentum operators acting on an eigenfuntion[3], ψ:

$$-i\frac{d\psi}{dx} \approx -i\frac{\psi(x_i + a) - \psi(x_i - a)}{2a} \tag{34}$$

[3]Units are chosen so that $c = \hbar = 1$.

$$i\frac{d^3\psi}{dx^3} \approx i\frac{\psi(x_i+3a) - 3\psi(x_i+a) + 3\psi(x_i-a) - \psi(x_i-3a)}{8a^3}. \tag{35}$$

These discrete approximations translate into the matricies,

$$-\frac{i}{2a}\begin{pmatrix} & & \vdots & & \\ \cdots & -1 & 0 & 1 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{36}$$

$$\frac{i}{8a^3}\begin{pmatrix} & & & & \vdots & & & & \\ \cdots & -1 & 0 & 3 & 0 & -3 & 0 & 1 & \cdots \\ & & & & \vdots & & & & \end{pmatrix} \tag{37}$$

Let $\mathbf{p}$ be the transverse momentum of the state. Define a state, $|\psi\rangle$, in operator form which, according to Bloch's theorem, must be an eigenstate of the P^r operator,

$$|\psi\rangle = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k,\mathbf{x})|0\rangle. \tag{38}$$

Thus, to find the eigenvalues of the P^r operator,

$$-\frac{i}{2a}\begin{pmatrix} & & \vdots & & \\ \cdots & -1 & 0 & 1 & \cdots \\ & & \vdots & & \end{pmatrix}\begin{pmatrix} \vdots \\ e^{i\mathbf{p}\cdot\mathbf{x}_n} \\ \vdots \end{pmatrix} = \lambda_1 \begin{pmatrix} \vdots \\ e^{i\mathbf{p}\cdot\mathbf{x}_n} \\ \vdots \end{pmatrix}. \tag{39}$$

Similarly, for the $(P^r)^3$ operator,

$$\frac{i}{8a^3}\begin{pmatrix} & & & & \vdots & & & & \\ \cdots & -1 & 0 & 3 & 0 & -3 & 0 & 1 & \cdots \\ & & & & \vdots & & & & \end{pmatrix}\begin{pmatrix} \vdots \\ e^{i\mathbf{p}\cdot\mathbf{x}_n} \\ \vdots \end{pmatrix} = \lambda_3 \begin{pmatrix} \vdots \\ e^{i\mathbf{p}\cdot\mathbf{x}_n} \\ \vdots \end{pmatrix}. \tag{40}$$

Solving for the eigenvalues of these equations shows,

$$\lambda_1 = \frac{1}{a}\sin(p^r a) \approx \frac{1}{a}\left(p^r a - \frac{(p^r a)^3}{6} + \frac{(p^r a)^5}{120}\right) \tag{41}$$

$$\lambda_3 = \frac{1}{4a^3}\left(-\sin(3p^r a) + 3\sin(p^r a)\right) \approx \frac{1}{a^3}\left((p^r a)^3 - \frac{(p^r a)^5}{2}\right) \tag{42}$$

This allows us to construct an "improved" version of P^r, one where finite lattice spacing errors are minimized. In the following, we will attempt to remove the leading error in the Taylor expansions given above. The linear combination

$$\lambda_{improved} = \lambda_1 + C\lambda_3 \tag{43}$$

removes this leading error if the λ_3 coefficient, $C = a^2/6$ is chosen. Thus,

$$\lambda_{improved} \approx \frac{1}{a}(p^r a + \frac{3(p^r a)^5}{40}) . \tag{44}$$

This improved eigenvalue can be used to construct an improved P^r in terms of creation annihilation operators. The final result is,

$$P^r = \frac{i}{2a}\sum_{\mathbf{x}}\int dk\,\left(a^\dagger(k,\mathbf{x}+\hat{r}a) - a^\dagger(k,\mathbf{x}-\hat{r}a)\right)a(k,\mathbf{x}) \tag{45}$$

$$(P^r)^3 = \frac{i}{8a^3}\sum_{\mathbf{x}}\int dk\big[\left(a^\dagger(k,\mathbf{x}-3\hat{r}a) - a^\dagger(k,\mathbf{x}+3\hat{r}a)\right)a(k,\mathbf{x})$$
$$+3\left(a^\dagger(k,\mathbf{x}+\hat{r}a) - a^\dagger(k,\mathbf{x}-\hat{r}a)\right)a(k,\mathbf{x})\big] \quad (46)$$

$$P^r_{improved} = P^r + C(P^r)^3, \quad (47)$$

where C is defined as above.

5.2 M_{-r} in the Transverse Lattice

Translating M_{-r} derived for the continuum into the transverse lattice means:

$$M_{-r} = \sum_{\mathbf{x}} x^r \int dk\, k\, a^\dagger(k,\mathbf{x})\, a(k,\mathbf{x}). \quad (48)$$

Applying a small boost, size $\varepsilon \ll 1$, to the state,

$$|C\rangle = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k,\mathbf{x})|0\rangle, \quad (49)$$

creates the boosted state

$$|E\rangle = |C\rangle + i\varepsilon k|V\rangle. \quad (50)$$

Where,

$$|V\rangle = \sum_{\mathbf{x}} x^r\, e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k,\mathbf{x})|0\rangle. \quad (51)$$

According to the algebra derived in Equation (28), measuring the momentum of a state boosted by M_{-r} should give,

$$P^r|E\rangle = (p^r + \varepsilon k)\,|C\rangle + i\varepsilon k p^r|V\rangle. \quad (52)$$

Because of the error in using a discrete approximation for M_{-r}, measuring the transverse momentum of $|E\rangle$ gives,

$$P^r|E\rangle = A(p^r)|C\rangle + \varepsilon k B(p^r)|C\rangle + i\varepsilon k\, A(p^r)|V\rangle. \quad (53)$$

Where

$$A(p^r) = \frac{9}{8a}\sin(p^r a) - \frac{1}{24a}\sin(3\,p^r a) \approx \frac{1}{a}\left((p^r a) - \frac{3(p^r a)^5}{30}\right) \quad (54)$$

and

$$B(p^r) = \frac{9}{8}\cos(p^r a) - \frac{1}{8}\cos(3\,p^r a) \approx 1 - \frac{3(p^r a)^4}{8}. \quad (55)$$

5.3 The Hamiltonian: P^-

Finding the Hamiltonian in the presence of a transverse lattice was accomplished by first finding the lattice approximation of $(\mathbf{P})^2$ in terms of the first two elements of its expasion:

$$(\mathbf{P})^2 \approx (P^1)^2 + (P^2)^2 + C((P^1)^4 + (P^2)^4), \quad (56)$$

where C is the coefficient given to minimize error introduced by finite lattice spaceing. The term, $(P^1)^2(P^2)^2$ will be ignored since it does not aid in minimizing this error. The discrete approximations of the second and forth derivatives of some function, ψ, are:

$$\frac{d^2\psi}{dx^2} \approx \frac{\psi(x_i+a) - 2\psi(x_i) + \psi(x_i-a)}{a^2}. \quad (57)$$

$$\frac{d^4\psi}{dx^4} \approx \frac{\psi(x_i+2a) - 4\psi(x_i+a) + 6\psi(x_i) - 4\psi(x_i-a) + \psi(x_i-2a)}{a^4}. \quad (58)$$

Including the corresponding factors of $-i$ (raised to the second or forth power), the matrices for these operators were found to be:

$$(P^r)^2 = -\frac{d^2\psi}{dx^2} \approx -\frac{1}{a^2}\begin{pmatrix} & & \vdots & & \\ \cdots & 1 & -2 & 1 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{59}$$

$$(P^r)^4 = \frac{d^4\psi}{dx^4} \approx \frac{1}{a^4}\begin{pmatrix} & & & \vdots & & & \\ \cdots & 1 & -4 & 6 & -4 & 1 & \cdots \\ & & & \vdots & & & \end{pmatrix} \tag{60}$$

Bloch's theorem suggests the form of the shared eigenvectors of these matrices is

$$|C\rangle = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k,\mathbf{x})|0\rangle. \tag{61}$$

The eigenvalue of the $(P^r)^2$ matrix is found to be:

$$\lambda_2 = \frac{-2}{a^2}(\cos(p^r a) - 1) \approx \frac{1}{a^2}\left((p^r a)^2 - \frac{(p^r a)^4}{12} + \frac{(p^r a)^6}{360}\right) \tag{62}$$

The eigenvalue for the $(P^r)^4$ matrix is found to be:

$$\lambda_4 = \frac{4}{a^4}(\cos^2(p^r a) - 2\cos(p^r a) + 1) \approx \frac{1}{a^4}\left((p^r a)^4 - \frac{(p^r a)^6}{6}\right) \tag{63}$$

The error introduced from finite lattice spacing is minimized when the improved eigenvalue is a linear combination of λ_2 and λ_4. Thus, the improved eigenvalue is

$$\lambda_{improved} = \lambda_2 + C\,\lambda_4, \tag{64}$$

where the coefficient C is chosen to be $a^4/12$. One can construct an *ansatz* for an improved approximation, $(\mathbf{P}^2)_{improved}$, to match this improved eigenvalue. It can be shown that the lattice approximation for $(P^r)^2$ is

$$(P^r)^2 = \frac{-2}{a^2}\sum_{\mathbf{x}}\left[\int dk\,\left(a^\dagger(k,\mathbf{x}+\hat{r}a) + a^\dagger(k,\mathbf{x}-\hat{r}a)\right)\frac{a(k,\mathbf{x})}{2} - a^\dagger(k,\mathbf{x})a(k,\mathbf{x})\right]. \tag{65}$$

The lattice approximation for $(P^r)^4$ is

$$(P^r)^4 = \frac{4}{a^4}\sum_{\mathbf{x}}\int dk\left[\left(a^\dagger(k,\mathbf{x}+2\hat{r}a) + a^\dagger(k,\mathbf{x}-2\hat{r}a)\right)\frac{a(k,\mathbf{x})}{4} - \left(a^\dagger(k,\mathbf{x}+\hat{r}a) + a^\dagger(k,\mathbf{x}-\hat{r}a)\right)a(k,\mathbf{x}) + a^\dagger(k,\mathbf{x})a(k,\mathbf{x})\frac{3}{2}\right]. \tag{66}$$

$(\mathbf{P})^2_{improved}$ is

$$(\mathbf{P})^2_{improved} = (P^r)^2 + \frac{a^2}{12}(P^r)^4. \tag{67}$$

Deriving P^- from this lattice approximation is accomplished simply by using Equation 1.7:

$$P^- = \frac{\mathbf{P}^2 + m^2}{2P^+}\ . \tag{68}$$

5.4 Error in Lattice Approximations

The following tables compare the error associated with the lattice approximation for each of the operators previously derived. In order to simplify notation, these results are tested on a state

$$|C\rangle = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} a^\dagger(k, \mathbf{x})|0\rangle. \tag{69}$$

Exact Commutators			
	M_{-r}	P^r	P^+P^-
P^+	0	0	0
M_{-r}	0	$iP^+\|C\rangle$	$-iP^+P^r\|C\rangle$
K_3	$iM_{-r}\|C\rangle$	0	0
P^r	$-iP^+\|C\rangle$	0	0

Commutators for Lattice Approximation			
	M_{-r}	P^r	P^+P^-
P^+	0	0	0
M_{-r}	0	$iP^+\left(1-\frac{3}{8}(\mathbf{p}\cdot\hat{r}b)^4\right. \| C\rangle$	$-iP^+\left((\mathbf{p}\cdot\hat{r}b)-\frac{1}{30}(\mathbf{p}\cdot\hat{r}b)^5\right)\frac{1}{b}\|C\rangle$
K_3	$iM_{-r}\|C\rangle$	0	0
P^r	$-iP^+\left(1-\frac{3}{8}(\mathbf{p}\cdot\hat{r}b)^4\right)\|C\rangle$	0	0

Thus, P^r and P^+P^- can always be improved by increasing the terms in their Taylor expansions. However, M_{-r} cannot be improved. Its form in the presence of the transverse lattice is exact.

6 The Femto-Worm Model

Matrices representing the Femto-Worm Model for the Hamiltonian operator, P^-, in QCD will be constructed for meson and baryon states in the transverse lattice. Let $b^\dagger(\mathbf{x})|0\rangle$ create a quark at $\mathbf{x}$, $d^\dagger(\mathbf{x})|0\rangle$ create an anti-quark at $\mathbf{x}$, and $a^\dagger_\lambda(\mathbf{x})|0\rangle$ create a gluon in the λ direction. These operators obey the following commutator and anti-commutator relations:

$$\begin{aligned} \left[a(\mathbf{x}), a^\dagger(\mathbf{y})\right] &= \delta_{\mathbf{x},\mathbf{y}} \\ \left\{b(\mathbf{x}), b^\dagger(\mathbf{y})\right\} &= \delta_{\mathbf{x},\mathbf{y}} \\ \left\{d(\mathbf{x}), d^\dagger(\mathbf{y})\right\} &= \delta_{\mathbf{x},\mathbf{y}} \\ \{a, d\} = \{a, b\} = \{b, d\} &= 0. \end{aligned} \tag{70}$$

6.1 Meson States

In the following section, P^- for mesons will be constructed. In order to simplify notation, only the variables $\mathbf{x}$ and $\mathbf{p}$ representing position and momentum in the transverse directions will be used. Defining the states,

$$
\begin{aligned}
|\mathbf{p}\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} b^\dagger(\mathbf{x})\, d^\dagger(\mathbf{x})|0\rangle && (71)\\
|1,\mathbf{p}\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}(\mathbf{x}+\hat{1}a/2)}\, b^\dagger(\mathbf{x})\, a^\dagger(\mathbf{x})\, d^\dagger(\mathbf{x}+\hat{1}a)|0\rangle, && (72)
\end{aligned}
$$

gives the basis

$$|\psi\rangle = C_1|\mathbf{p}\rangle + C_2|1,\mathbf{p}\rangle. \tag{73}$$

In operator form, the Femto-Worm approximation of the Hamiltonian for mesons is

$$
\begin{aligned}
P^- = \sum_{\mathbf{x},\lambda} \Big[& {m_g}^2\, a_\lambda^\dagger(\mathbf{x}) a_\lambda(\mathbf{x}) + {m_q}^2 \big(b^\dagger(\mathbf{x})\, b(\mathbf{x}) - d(\mathbf{x})\, d^\dagger(\mathbf{x})\big)\\
& +\kappa\big(b^\dagger(\mathbf{x})\, {a_\lambda}^\dagger(\mathbf{x})\, b(\mathbf{x}+\hat{\lambda}a) + b^\dagger(\mathbf{x}+\hat{\lambda}a)\, a_\lambda(\mathbf{x})\, b(\mathbf{x})\\
& +d(\mathbf{x})\, {a_\lambda}^\dagger(\mathbf{x})\, d^\dagger(\mathbf{x}+\hat{\lambda}a) + d(\mathbf{x}+\hat{\lambda}a)\, a_\lambda(\mathbf{x})\, d^\dagger(\mathbf{x})\big)\Big].
\end{aligned} \tag{74}
$$

Thus, the operator form of the hamiltonian is

$$
\begin{pmatrix} \langle\mathbf{p}|P^-|\mathbf{p}\rangle & \langle\mathbf{p}|P^-|1,\mathbf{p}\rangle \\ \langle 1,\mathbf{p}|P^-|\mathbf{p}\rangle & \langle 1,\mathbf{p}|P^-|1,\mathbf{p}\rangle \end{pmatrix} = \begin{pmatrix} 2m_q^2 & \kappa(e^{-ip^1a/2} - e^{ip^1a/2}) \\ \kappa(e^{ip^1a/2} - e^{-ip^1a/2}) & m_g^2 + 2m_q^2 \end{pmatrix} \tag{75}
$$

One can find the eigenvalues of this matrix to be

$$\lambda = \frac{4m_q^2 + m_g^2 \pm \sqrt{m_g^4 - 8\kappa^2\cos(p^1a) + 8\kappa^2}}{2} \approx \frac{4m_q^2 + m_g^2}{2} \pm \left(\frac{m_g^2}{2} + \frac{\kappa^2}{m_g^2}(p^1a)^2\right). \tag{76}$$

Thus, we can get the desired P^- verses P^1 behavior by tuning κ. Only one state will be correct. The other will be tachyonic.

6.2 Baryon

In the following section, we will consider a toy model for baryons in order to derive its Hamiltonian. In addition to position and momentum, color vectors will be introduced as latin alphabetic subscripts. Although the number of colors N is 3, we will keep it algebraic for clarity. We begin by defining the following three-quark states.

$$
\begin{aligned}
|1\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{6N!}} \epsilon_{ijk}\, b_i^\dagger(\mathbf{x})\, b_j^\dagger(\mathbf{x})\, b_k^\dagger(\mathbf{x})|0\rangle\\
|2\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2N!N}} \epsilon_{ijk} \left(b^\dagger(\mathbf{x}-\hat{1}a)\, a_1^\dagger(\mathbf{x}-\hat{1}a)\right)_i b_j^\dagger(\mathbf{x})\, b_k^\dagger(\mathbf{x})|0\rangle\\
|3\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2N!N^2}} \epsilon_{ijk} \left(b^\dagger(\mathbf{x}-\hat{1}a)\, a_1^\dagger(\mathbf{x}-\hat{1}a)\right)_i \left(b^\dagger(\mathbf{x})\, a_1^\dagger(\mathbf{x}-\hat{1}a)\right)_j b_k^\dagger(\mathbf{x})|0\rangle\\
|4\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2N!N}} \epsilon_{ijk} \left(b^\dagger(\mathbf{x}+\hat{1}a)\, a_{-1}^\dagger(\mathbf{x}+\hat{1}a)\right)_i b_j^\dagger(\mathbf{x})\, b_k^\dagger(\mathbf{x})|0\rangle\\
|5\rangle &= \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2N!N^2}} \epsilon_{ijk} \left(b^\dagger(\mathbf{x}+\hat{1}a)\, a_{-1}^\dagger(\mathbf{x}+\hat{1}a)\right)_i \left(b^\dagger(\mathbf{x}+\hat{1}a)\, a_{-1}^\dagger(\mathbf{x}+\hat{1}a)\right)_j b_k^\dagger(\mathbf{x})|0\rangle.
\end{aligned} \tag{77}
$$

Thus, our basis will be,

$$|\psi\rangle = C_1|1\rangle + C_2|2\rangle + C_3|3\rangle + C_4|4\rangle + C_5|5\rangle. \tag{78}$$

In operator form, the hamiltonian for baryons will be,

$$\begin{aligned} P^- &= \sum_{\mathbf{x},\lambda} \Bigg[{m_g}^2\, a^\dagger(\mathbf{x})a(\mathbf{x}) + {m_q}^2 b^\dagger(\mathbf{x})b(\mathbf{x}) \\ &+ \kappa \left(b^\dagger(\mathbf{x}){a_\lambda}^\dagger(\mathbf{x})b(\mathbf{x}+\hat{\lambda}a) + b^\dagger(\mathbf{x}+\hat{\lambda}a)a_\lambda(\mathbf{x})b(\mathbf{x}) \right) \\ &+ \xi\, \epsilon_{abc}\, \epsilon_{def} \Bigg(\left(a^\dagger_\lambda(\mathbf{x})\right)_{ad} \left(a^\dagger_\lambda(\mathbf{x})\right)_{be} \left(a_{-\lambda}(\mathbf{x}+\hat{\lambda}a)\right)_{cf} \\ &+ \left(a_\lambda(\mathbf{x})\right)_{ad} \left(a_\lambda(\mathbf{x})\right)_{be} \left(a^\dagger_{-\lambda}(\mathbf{x}+\hat{\lambda}a)\right)_{cf} \Bigg) \Bigg]. \end{aligned} \tag{79}$$

Thus, the P^- matrix will be

$$\begin{aligned} P^- &= \begin{pmatrix} \langle 1|P^-|1\rangle & \langle 1|P^-|2\rangle & \langle 1|P^-|3\rangle & \langle 1|P^-|4\rangle & \langle 1|P^-|5\rangle \\ \langle 2|P^-|1\rangle & \langle 2|P^-|2\rangle & \langle 2|P^-|3\rangle & \langle 2|P^-|4\rangle & \langle 2|P^-|5\rangle \\ \langle 3|P^-|1\rangle & \langle 3|P^-|2\rangle & \langle 3|P^-|3\rangle & \langle 3|P^-|4\rangle & \langle 3|P^-|5\rangle \\ \langle 4|P^-|1\rangle & \langle 4|P^-|2\rangle & \langle 4|P^-|3\rangle & \langle 4|P^-|4\rangle & \langle 4|P^-|5\rangle \\ \langle 5|P^-|1\rangle & \langle 5|P^-|2\rangle & \langle 5|P^-|3\rangle & \langle 5|P^-|4\rangle & \langle 5|P^-|5\rangle \end{pmatrix} \\ &= \begin{pmatrix} 3m_q^2 & \sqrt{3N}\kappa & 0 & \sqrt{3N}\kappa & 0 \\ \sqrt{3N}\kappa & m_g^2 + 3m_q^2 & 2\sqrt{N}\kappa & 0 & \xi\frac{2N!}{N^{\frac{3}{2}}}e^{-ip^1a} \\ 0 & 2\sqrt{N}\kappa & 2m_g^2 + 3m_q^2 & \xi\frac{2N!}{N^{\frac{3}{2}}}e^{-ip^1a} & 0 \\ \sqrt{3N}\kappa & 0 & \xi\frac{2N!}{N^{\frac{3}{2}}}e^{ip^1a} & m_g^2 + 3m_q^2 & 2\sqrt{N}\kappa \\ 0 & \xi\frac{2N!}{N^{\frac{3}{2}}}e^{ip^1a} & 0 & 2\sqrt{N}\kappa & 2m_g^2 + 3m_q^2 \end{pmatrix}. \end{aligned} \tag{80}$$

The coupling constants can be chosen so that the lowest energy state has a quadratic dispersion relation and there is an approximately equal probability for each state. One choice of coefficients for this to work is

$$m_g = 1,\ m_q = 3.5,\ \kappa = 5.25,\ \xi = 5.25. \tag{81}$$

If ξ is positive, the dispersion relation for the lowest energy state is concave up. If ξ is negative, the dispersion relation for the lowest energy state is concave down. Figure 3 shows the dispersion relation for the lowest energy state of the Hamiltonian. Figure 4 shows the dispersion relation for all states.

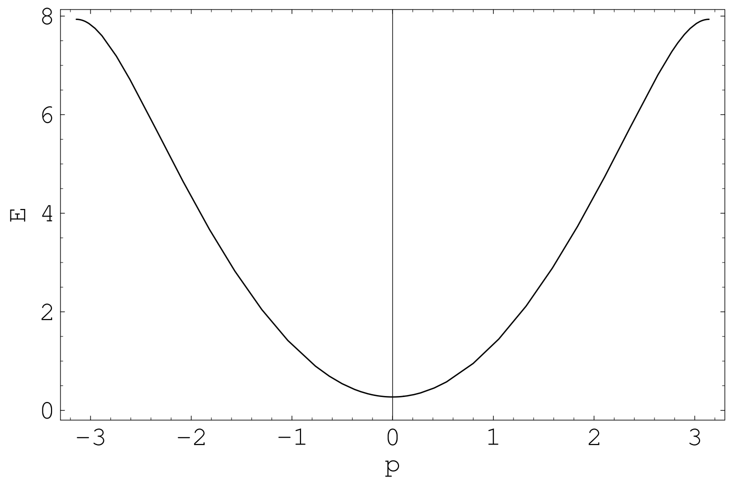

Figure 3: Energy of the lowest eigenstate versus momentum.

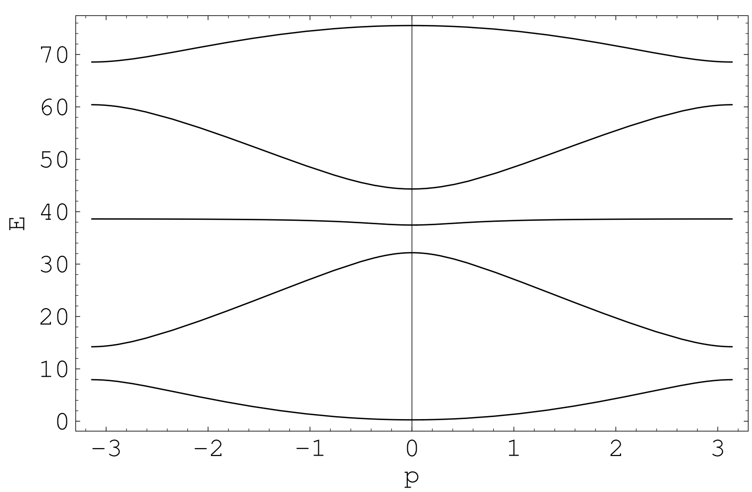

Figure 4: All eigenstates versus momentum.